Exponential Riesz bases of subspaces and divided differences [1]

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Abstract

Linear combinations of exponentials $e^{i\lambda_k t}$ in the case where the distance between some points λ_k tends to zero are studied. D. Ullrich [30] has proved the basis property of the divided differences of exponentials in the case when $\{\lambda_k\} = \bigcup \Lambda^{(n)}$ and the groups $\Lambda^{(n)}$ consist of equal number of points all of them are close enough to n, $n \in \mathbf{Z}$. We have generalized this result for groups with arbitrary number of close points and obtained a full description of Riesz bases of exponential divided differences.

1 Introduction

Families of 'nonharmonic' exponentials $\left\{e^{i\lambda_k t}\right\}$ appear in various fields of mathematics such as the theory of nonselfadjoint operators (Sz.-Nagy–Foias model), the Regge problem for resonance scattering, the theory of linear initial boundary value problems for partial differential equations, control theory for distributed parameter systems, and signal processing. One of the central problems arising in all of these applications is the question of the Riesz basis property of an exponential family. In the space $L^2(0,T)$ this problem was considered for the first time in the classical work of R. Paley and N. Wiener [27], and since then has motivated a great deal of work by many mathematicians; a number of references are given in [18], [34] and [2]. The problem was ultimately given a complete solution [26], [18], [23] on the basis of an approach suggested by B. Pavlov.

The main result in this direction can be formulated as follows [26].

Theorem 1 *Let* $\Lambda := \{\lambda_k | k \in \mathbb{Z}\}$ *be a countable set of the complex plane. The family* $\left\{e^{i\lambda_k t}\right\}$ *forms a Riesz basis in* $L^2(0,T)$ *if and only if the following conditions are satisfied:*

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(i) Λ lies in a strip parallel to the real axis,

$$\sup_{k\in\mathbb{Z}} |\Im\lambda_k| < \infty\,,$$

and is uniformly discrete (or separated), i.e.

$$\delta(\Lambda) := \inf_{k\neq n} |\lambda_k - \lambda_n| > 0\,; \tag{1}$$

(ii) there exists an entire function F of exponential type with indicator diagram of width T and zero set Λ (the generating function of the family $\left\{e^{i\lambda_k t}\right\}$ on the interval $(0,T)$) *such that, for some real h, the function $|F(x+ih)|^2$ satisfies the Helson–Szegö condition: functions $u, v \in L^\infty(\mathbb{R})$, $\|v\|_{L^\infty(\mathbb{R})} < \pi/2$ may be found such that*

$$|F(x+ih)|^2 = \exp\{u(x) + \tilde{v}(x)\} \tag{2}$$

Here the map $v \mapsto \tilde{v}$ denotes the Hilbert transform for bounded functions:

$$\tilde{v}(x) = \frac{1}{\pi}\int_{-\infty}^{\infty} v(t)\left\{\frac{1}{x-t} - \frac{t}{t^2+1}\right\} dt.$$

Remark 1 *Note that (i) is equivalent to Riesz basis property of $\mathcal{E}$ in its span in $L^2(0,\infty)$ and (ii) is a criterion that the orthoprojector P_T from this span into $L^2(0,T)$ is an isomorphism (bounded and boundedly invertible operator).*

It is well known that the Helson–Szegö condition is equivalent to the Muckenhoupt condition (A_2):

$$\sup_{I\in\mathcal{J}}\Big\{\frac{1}{|I|}\int_I |F(x+ih)|^2\, dx\ \frac{1}{|I|}\int_I |F(x+ih)|^{-2}\, dx\Big\}\ <\ \infty,$$

where $\mathcal{J}$ is the set of all intervals of the real axis.

The notion of the generating function mentioned above plays a central role in the modern theory of nonharmonic Fourier series [18, 2]. This notion plays also an important role in the theory of exponential bases in Sobolev spaces (see [3, 16, 21]). It is possible to write the explicit expression for this function

$$F(z) = \lim_{R\to\infty} \prod_{|\lambda_k|\leq R} (1 - \frac{z_k}{\lambda_k})$$

(we replace the term $(1-\lambda_k^{-1}z)$ by z if $\lambda_k = 0$).

The theory of nonharmonic Fourier series was successfully applied to control problems for distributed parameter systems and formed the base of the

powerful method of moments ([11, 28, 2]). Recent investigations into new classes of distributed systems such as hybrid systems, structurally damped systems have raised a number of new difficult problems in the theory of exponential families (see, e.g. [15, 24, 17]). One of them is connected with the properties of the family $\mathcal{E} = \left\{e^{i\lambda_k t}\right\}$ in the case when the set Λ does not satisfy the separation condition (1), and therefore $\mathcal{E}$ does not form a Riesz basis in its span in $L^2(0,T)$ for any $T > 0$.

Properties of such families in $L^2(0,\infty)$ have been studied for the first time in the paper of V. Vasyunin [31] (see also [25, Lec. IX]). In the case when Λ is a finite union of separated sets, a natural way to represent Λ as a set of groups $\Lambda^{(p)}$ of close points was suggested. The subspaces spanned on the corresponding exponentials form a Riesz basis. This means that there exists an isomorphism mapping these subspaces into orthogonal ones. This fact together with Pavlov's result on the orthoprojector P_T (see Remark above) gives a criterion of the Riesz basis property of subspaces of exponentials in $L^2(0,T)$: the generating function have to satisfy the Helson–Szegö condition (2). Note that for the particular case when the generating function is a sine type function (see definition in [20], [18], [2]), theorem of such a kind was proved by Levin [20].

Thus, Vasyunin's result and Pavlov's geometrical approach give us description of exponential Riesz bases of subspaces. If we do have a Riesz basis of subspaces, clearly, we can choose an orthonormalized basis in each subspace and obtain a Riesz basis of *elements*. However, this way is not convenient in applications when we need more explicit formulae. It is important to obtain description of Riesz bases of elements which are 'simple and natural' linear combinations of exponentials.

The first result in this direction was obtained by D. Ullrich [30] who considered sets Λ of the form $\Lambda = \bigcup_{n\in\mathbb{Z}} \Lambda^{(n)}$, where subsets $\Lambda^{(n)}$ consist of equal number (say, N) real points $\lambda_1^{(n)}, \ldots, \lambda_N^{(n)}$ close to n, i.e., $|\lambda_j^{(n)} - n| < \varepsilon$ for all j and n. He proved that for sufficiently small $\varepsilon > 0$ (no estimate of ε was given) the family of particular linear combinations of exponentials $e^{i\lambda_k t}$ — the so–called *divided differences* constructed by subsets $\Lambda^{(n)}$ (see Definition 1 in subsection 2.2) — forms a Riesz basis in $L^2(0, 2\pi N)$. Such functions arise in numerical analysis [29], and the divided difference of $e^{i\mu t}$, $e^{i\lambda t}$ of the first order is $(e^{i\mu t} - e^{i\lambda t})/(\mu - \lambda)$. In a sense, the Ullrich result may be considered as a perturbation theorem for the basis family $\left\{e^{int}, te^{int}, \ldots, t^{N-1}e^{int}\right\}$, $n \in \mathbf{Z}$.

The conditions of this theorem are rather restrictive and it can not be applied to some problems arising in control theory (see, e.g. [8, 12, 17, 22, 24]).

In the present paper we generalize Ullrich's result in several directions:

the set Λ is allowed to be complex, subsets $\Lambda^{(n)}$ are allowed to contain an arbitrary number of points, which are not necessarily 'very' close to each other (and, moreover, to some integer).

Actually, we give a full description of Riesz bases of exponential divided differences and generalized divided differences (the last ones appear in the case of multiple points λ_n). To be more specific, we take a *sequence* Λ which is 'a union' of a finite number of separated sets. Following Vasyunin we decompose Λ into groups $\Lambda^{(p)}$, then choose for each group the family of the generalized divided differences (GDD) and prove that these functions form a Riesz basis in $L^2(0,T)$ if the generating function of the exponential family satisfies the Helson–Szegö condition (2). To prove that we show that GDD for points $\lambda_1, \dots, \lambda_N$ lying in a fixed ball form 'a uniform basis', i.e. the basis constants do not depend on the positions of λ_j in the ball. Along with that, GDD depend on parameters analytically. Thus, this family is a natural basis for the situation when exponentials $e^{i\lambda t}, \lambda \in \Lambda$ do not form even uniformly minimal family. For the particular case $\Lambda = \{n\alpha\} \cup \{n\beta\}, n \in \mathbf{Z}$, appearing in a problem of simultaneous control, this scheme was realized in [7].

In the case when Λ is not a finite union of separated sets, we present a negative result: for some ordering of Λ, GDDs do not form a uniformly minimal family.

Remark 2 *After this paper has been written, a result on Riesz bases of exponential DD in their span in $L^2(0,T)$ for large enough T has been announced in [9]. There, though Λ is contained in $\mathbb{R}$. For more general results in this direction see [5, 6].*

Remark 3 *In a series of papers [32], [33], [10], [13], [14]), the free interpolation problem has been studied and a description of traces of bounded analytic functions on a finite union of Carleson sets has been obtained in terms of divided differences. In view of well known connections between interpolation and basis properties, these results may be partially ([13], [14]) rewritten in terms of geometrical properties of exponential DD in $L^2(0,\infty)$.*

2 Main Results

Let $\Lambda = \{\lambda_n\}$ be a sequence in $\mathbb{C}$ ordered in such a way that $\Re\lambda_n$ form a nondecreasing sequence. We connect with Λ the exponential family

$$\mathcal{E}(\Lambda) = \{e^{i\lambda_n t}, te^{\lambda_n t}, \dots, t^{m_{\lambda_n}-1}e^{i\lambda_n t}\},$$

where m_{λ_n} is the multiplicity of $\lambda_n \in \Lambda$.

For the sake of simplicity, we confine ourselves to the case $\sup |\Im \lambda_n| < \infty$. The multiplication operator $f(t) \mapsto e^{-at} f(t)$ is an isomorphism in $L^2(0,T)$ for any T and maps exponential functions $e^{i\lambda_n t}$ to $e^{i(\lambda_n + ia)t}$. Since we are interesting in the Riesz basis property of linear combinations of functions $t^r e^{i\lambda_n t}$ in $L^2(0,T)$, we can suppose without loss of generality that Λ lies in a strip $S := \{z |\, 0 < \alpha \le \Im\lambda \le \beta < \infty\}$ in the upper half plane.

The sequence Λ is called *uniformly discrete* or *separated* if condition (1) is fulfilled. Note that in this case all points λ are simple and we do not need differentiate between a sequence and a set.

We say that Λ *relatively uniformly discrete* if Λ can be decomposed into a finite number of uniformly discrete subsequences. Sometimes we shall simply say that such a Λ is a finite union of uniformly discrete sets, however we always consider a point λ_n to be assigned a multiplicity.

2.1 Splitting of the spectrum and subspaces of exponentials

Here we introduce notations needed to formulate the main result. For any $\lambda \in \mathbb{C}$, denote by $D_\lambda(r)$ a disk with center λ and radius r. Let $G^{(p)}(r)$, $p = 1, 2, \dots,$ be the connected components of the union $\cup_{\lambda \in \Lambda} D_\lambda(r)$. Write $\Lambda^{(p)}(r)$ for the subsequences of Λ lying in $G^{(p)}$, $\Lambda^{(p)}(r) := \{\lambda_n |\, \lambda_n \in G^{(p)}(r)\}$, and $\mathcal{L}^{(p)}(r)$ for subspaces spanned by corresponding exponentials $\{t^n e^{i\lambda t}\}$, $\lambda \in \Lambda^{(p)}(r), n = 0, \dots, m_\lambda - 1$.

Lemma 1 *Let Λ be a union of N uniformly discrete sets Λ_j,*

$$\delta_j := \delta(\Lambda_j) := \inf_{\lambda \ne \mu;\ \lambda, \mu \in \Lambda_j} |\lambda - \mu|,\ \ \delta := \min_j \delta_j.$$

Then for

$$r < r_0 := \frac{\delta}{2N}$$

the number $\mathcal{N}^{(p)}(r)$ of elements of $\Lambda^{(p)}$ is at most N.

PROOF: Let points μ_k, $k = 1, \dots, N+1$, belong to the same $\Lambda^{(p)}$. Then the distance between any two of these points is less than $2rN$ and so less than δ. From the other hand, there at least two among $N+1$ points which belong to the same Λ_j and, therefore, the distance between them is not less than δ. This contradiction proves the lemma. ■

We call by *$\mathcal{L}$-basis* a family in a Hilbert space which forms a Riesz basis in the closure of its linear span.

The following statement is a small modification of the theorem of Vasyunin [31].

Lemma 2 *Let Λ be a relatively uniformly discrete sequence. Then for any $r > 0$ the family of subspaces $\mathcal{L}^{(p)}(r)$ forms an $\mathcal{L}$-basis in $L^2(0,\infty)$.*

This statement is proved in [31] and [25, Lect. IX] for $r = r_0/16$ and for disks in so called hyperbolic metrics, however one can easily check that the proof with obvious modifications remains valid for all r.

In applications we often meet the case of real $\Lambda's$. Then a relatively discrete set can be characterized using another parameter than in Lemma 1. The following statement can be easily proved similar to Lemma 1.

Lemma 3 *A real sequence Λ is a union of N uniformly discrete sets Λ_j if and only if $\inf_n(\lambda_{n+N} - \lambda_n) := \tilde{\delta} > 0$ for all n. Along with that, $\min_j \delta(\Lambda_j) \leq \tilde{\delta}$*

2.2 Divided differences

Let μ_k, $k = 1, \ldots, m$, be arbitrary complex numbers, not necessarily distinct.

Definition 1 *Generalized divided difference (GDD) of order zero of $e^{i\mu t}$ is $[\mu_1](t) := e^{i\mu_1 t}$. GDD of the order $n-1$, $n \leq m$, of $e^{i\mu t}$ is*

$$[\mu_1, \ldots, \mu_n] := \begin{cases} \dfrac{[\mu_1, \ldots, \mu_{n-1}] - [\mu_2, \ldots, \mu_n]}{\mu_1 - \mu_n}, & \mu_1 \neq \mu_n, \\ \dfrac{\partial}{\partial \mu}[\mu, \mu_2, \ldots, \mu_{n-1}]\Big|_{\mu=\mu_1}, & \mu_1 = \mu_n. \end{cases}$$

If all μ_k are distinct one can easily derive the explicit formulae for GDD:

$$[\mu_1, \ldots, \mu_n] = \sum_{k=1}^{n} \frac{e^{i\mu_k t}}{\prod_{j\neq k}(\mu_k - \mu_j)}. \tag{3}$$

For any points $\{\mu_k\}$ we can write [29, p. 228]

$$\begin{aligned} [\mu_1, \ldots, \mu_n] = \int_0^1 d\tau_1 \int_0^{\tau_1} d\tau_2 \ldots \int_0^{\tau_{n-2}} d\tau_{n-1} (it)^{n-1} \\ \exp\Big(it\,[\mu_1 + \tau_1(\mu_2 - \mu_1) + \ldots + \tau_{n-1}(\mu_n - \mu_{n-1})]\Big). \end{aligned} \tag{4}$$

Theorem 2 *The following statements are true.*

(i) Functions $\varphi_1 := [\mu_1], \ldots, \varphi_n := [\mu_1, \ldots, \mu_n]$, depend on parameters μ_j continuously and symmetrically. If points $\mu_1, \ldots, \mu_n$ are in a convex domain $\Omega \subset \mathbb{C}$, then

$$|\varphi_j(t)| \leq c_n e^{\gamma t}, \gamma := -\inf_{z\in\Omega} \Im z. \tag{5}$$

(ii) Functions $\varphi_1, \ldots, \varphi_n$, are linearly independent.

(iii) If points $\mu_1, \ldots, \mu_n$ are distinct, then the family of GDD's forms a basis in the span of exponentials $e^{i\mu_1 t}, \ldots, e^{i\mu_n t}$.

(iv) Translation of the set $\mu_1, \ldots, \mu_n$ leads to multiplying of GDD's by an exponential:

$$[\mu_1 + \lambda, \ldots, \mu_n + \lambda] = e^{i\lambda t}[\mu_1, \ldots, \mu_n].$$

(v) For any $\varepsilon > 0$ and any $N \in I\!N$, there exists δ such that the estimates

$$||[\mu_1, \ldots, \mu_j](t) - e^{i\mu t}t^{j-1}/(j-1)!||_{L^2(0,\infty)} < \varepsilon,\ j = 1, \ldots, N,$$

are valid for any μ in the strip S and all points $\mu_1, \ldots, \mu_N$ belonging to the disk $D_\mu(\delta)$ of radius δ with the center at μ.

2.3 Bases of elements

Let $\Lambda^{(p)}(r) = \{\lambda_{j,p}\}, j = 1, \ldots, \mathcal{N}^{(p)}(r)$ be subsequences described in subsection 2.1 Denote by $\{\mathcal{E}^{(p)}(\Lambda, r)\}$ the family of GDD corresponding to the points $\Lambda^{(p)}(r)$:

$$\mathcal{E}^{(p)}(\Lambda, r) = \{[\lambda_{1,p}], [\lambda_{1,p}, \lambda_{2,p}], \ldots, [\lambda_{1,p}, \ldots, \lambda_{\mathcal{N}^{(p)},p}]\}.$$

Note that $\mathcal{E}^{(p)}(\Lambda, r)$ depends on enumeration of $\Lambda^{(p)}(r)$, although every GDD depends symmetrically on its parameters, see the assertion (i) of the last theorem.

Theorem 3 *Let Λ be a relatively uniformly discrete sequence and $r < r_0$. Then*

(i) the family $\{\mathcal{E}^{(p)}(\Lambda, r)\}$ forms a Riesz basis in $L^2(0, T)$ if and only if there exists an entire function F of exponential type with indicator diagram of width T and zeros at points λ_n multiplicity m_{λ_n} (the generating function of the family $\mathcal{E}(\Lambda)$ on the interval $(0, T)$) *such that, for some real h, the function $|F(x + ih)|^2$ satisfies the Helson–Szegö condition (2);*

(ii) for any finite sequence $\{a_{p,j}\}$ the inequality

$$\|\sum_{p,j} a_{p,j}e^{i\lambda_{j,p}t}\|^2_{L^2(0,T)} \geq C\sum_{p,j} |a_{p,j}|^2 \delta_p^{2(\mathcal{N}^{(p)}(r)-1)}$$

is valid with a constant C independent of $\{a_{p,j}\}$, where

$$\delta_p := \min\{|\lambda_{j,p} - \lambda_{k,p}| \,\big|\, k \neq j\}.$$

Suppose now that Λ is not a relatively uniformly discrete sequence. Then,

$$\sup_p \mathcal{N}^{(p)}(r) = \infty \quad \text{for any} \quad r > 0.$$

It is possible also that there is an infinite set $\Lambda^{(p)}$.

We show that in this case the family of GDDs is not uniformly minimal even in $L^2(0,\infty)$ at least, for some enumeration of points.

Theorem 4 *For any $r > 0$, there exists numbering of points in the subsequences $\Lambda^{(p)}$ such that family $\{\mathcal{E}^{(p)}(\Lambda, r)\}$ is not uniformly minimal in $L^2(0,\infty)$. Moreover, for any ε, there exists a pair φ_m, φ_{m+1} of GDD, corresponding to some set $\Lambda^{(p)}(r)$ such that*

$$\text{angle}_{\,L^2(0,\infty)}(\varphi_m, \varphi_{m+1}) < \varepsilon.$$

2.4 Application to an observation problem

Before starting the proofs of main results, we present an application of Theorem 3 to observability of a coupled 1d system studied in [19]:

$$\begin{cases} \frac{\partial^2}{\partial t^2} u_1 - \frac{\partial^2}{\partial x^2} u_1 + A u_1 + B u_2 = 0 & \text{in } (0,\pi) \times \mathbb{R}, \\ \frac{\partial^2}{\partial t^2} u_2 + \frac{\partial^4}{\partial x^4} u_2 + C u_1 + D u_2 = 0 & \text{in } (0,\pi) \times \mathbb{R}, \\ u_1 = u_2 = \frac{\partial^2}{\partial x^2} u_2 = 0 & \text{for } x = 0, \pi, \\ u_1 = y_0 \in H_0^1(0,\pi), \ \frac{\partial}{\partial t} u_1 = y_1 \in L^2(0,\pi), & \text{for } t = 0, \\ u_2 = \frac{\partial}{\partial t} u_2 = 0 & \text{for } t = 0 \end{cases}$$

(A, B, C, D are constants).

We introduce the initial energy E_0 of the system, $E_0 := \|y_0\|_{H^1}^2 + \|y_1\|_{L^2}^2$.

In the paper of C. Baiocchi, V. Komornik, and P. Loreti [19] the partial observability, i.e. inequality

$$\left\| \frac{\partial}{\partial x} u_1(0,t) \right\|_{L^2(0,T)}^2 \geq c E_0 \tag{6}$$

with a constant c independent of y_0 and y_1, has been proved for almost all 4–tuples (A, B, C, D) and for $T > 4\pi$. (It means that we can recover the initial state via the observation $\left\| \frac{\partial}{\partial x} u_1(0,t) \right\|_{L^2(0,T)}^2$ during the time T and the operator: *observation $\to$ initial state* is bounded. The authors conjectured the system is probably partially observed for $T > 2\pi$. Here we demonstrate this fact using the basis property of exponential DD.

Proposition 1 *For almost all 4–tuples (A, B, C, D) and for $T > 2\pi$ the estimate (6) is valid.*

To prove this proposition we use the representation and properties of the solution given in the paper [19]. To apply the Fourier method we introduce the eigenfrequencies $\omega_k,\ \nu_k,\ k \in \mathbb{N}$ of the system, where ν_k^2 and ω_k^2 are the eigenvalues of the matrix

$$\begin{pmatrix} k^2 + A & B \\ C & k^4 + D \end{pmatrix}.$$

It is easy to see that the following asymptotic relations are valid:

$$\nu_k = k + A/2k + O(k^{-3}),\ \omega_k = k^2 + D/2k^2 + O(k^{-6}). \tag{7}$$

We suppose that all ω_k and ν_k are distinct (this is true for almost all 4-tuples). Then the first component of the solution of the system can be written in the form

$$u_1(x,t) = \sum_{k \in \mathbf{K}} \left[\alpha_k e^{i\omega_k t} + \beta_k e^{i\nu_k t} \right] \sin kx, \tag{8}$$

where $\mathbf{K} := \mathbb{Z} \backslash \{0\},\ \omega_{-k} := -\omega_k,\ \nu_{-k} := -\nu_k,$

The coefficients $\alpha_k,\ \beta_k$ entered the last sum can be expressed via the initial data, and the authors of [19] show that under zero initial condition for u_2, we have

$$|\alpha_k|^2 + |\alpha_{-k}|^2 \prec k^{-8}(|\beta_k|^2 + |\beta_{-k}|^2) \tag{9}$$

and

$$E_0 \asymp \sum_{k \in \mathbf{K}} k^2 |\beta_k|^2. \tag{10}$$

Relations (9) and (10) mean, correspondingly, one–sided and two–sided inequalities with constants which do not depend on sequences $\{\alpha_k\}$ and $\{\beta_k\}$.

We now have to study the exponential family

$$\mathcal{E} = \{e^{i\lambda t}\}_{\lambda \in \Lambda},\ \Lambda = M \cup \Omega,\ M = \{\nu_k\},\ \Omega = \{\omega_k\},\ k \in \mathbf{K}.$$

For $r < 1$ and for large enough $|\lambda|$ the family $\Lambda^{(p)}(r)$ consist of one point ν_k if k is not a full square or of two points $\nu_{\operatorname{sign} k\, k^2}$ and ω_k. Denote by $\mathcal{E}^{(p)}(\Lambda, r)$ the related family of exponential DD.

Suppose for a moment that there exists an entire function F of exponential type with indicator diagram of width $T > 2\pi$ vanishing at Λ (F may also have another zeros) such that $|F(x+ih)|^2$ satisfies the Helson–Szegö condition (2) for some real h. Then, by Theorem 3, the family $\mathcal{E}^{(p)}(\Lambda, r)$ forms an $\mathcal{L}$-basis in $L^2(0,T)$. Expanding exponentials in DD of the zero and first order,

$$\alpha e^{i\lambda t} + \beta e^{i\mu t} = (\alpha + \beta) e^{i\lambda t} - \beta(\lambda - \mu)[\lambda, \mu],$$

for finite sequences $\{p_k\}$, $\{q_k\}$ we obtain the estimate

$$\left\|\sum_{k\in\mathbf{K}}\left[p_k e^{i\omega_k t}+q_k e^{i\nu_k t}\right]\right\|^2_{L^2(0,T)} \asymp \sum_{k\in\mathbf{K},\sqrt{|k|}\notin\mathbf{N}}|q_k|^2+ \tag{11}$$
$$\textstyle\sum_{k\in\mathbf{K}}\left[|p_k+q_{\operatorname{sign}k\,k^2}|^2+|\omega_k-\nu_{\operatorname{sign}k\,k^2}|^2|q_{\operatorname{sign}k\,k^2}|^2\right].$$

In the right hand side of this relation the first sum corresponds to one dimensional subspaces $\mathcal{L}^{(p)}(r)$ (this sum is taken for all k which are not full squares). The second sum corresponds to two dimensional subspaces. Together with (8), it follows

$$\left\|\frac{\partial}{\partial x}u_1(0,t)\right\|^2_{L^2(0,T)}=\left\|\sum_{k\in\mathbf{K}}k\left[\alpha_k e^{i\omega_k t}+\beta_k e^{i\nu_k t}\right]\right\|^2_{L^2(0,T)}$$
$$\succ\sum_{k\in\mathbf{K},\sqrt{|k|}\notin\mathbf{N}}k^2|\beta_k|^2+\sum_k\left[|\omega_k-\nu_{\operatorname{sign}k\,k^2}|^2|k^2\beta_{\operatorname{sign}k\,k^2}|^2+\left|k\alpha_k+k^2\beta_{\operatorname{sign}k\,k^2}\right|^2\right].$$

For R large enough using (9) we have

$$\sum_{|k|>R}\left[k\alpha_k+k^2\beta_{\operatorname{sign}k\,k^2}\right]\succ\sum_{|k|>R}k^2|\beta_{\operatorname{sign}k\,k^2}|^2.$$

Thus, we obtain

$$\left\|\frac{\partial}{\partial x}u_1(0,t)\right\|^2_{L^2(0,T)}\succ\sum_{k\in\mathbf{K}}k^2|\beta_k|^2\asymp E_0.$$

In order to complete the proof of Proposition 1, it remains to construct the function F. We do that using the following proposition which is interesting in its own right.

Proposition 2 *Let $\Lambda=\{\lambda_n\}$ be a zero set of a sine type function (see definition in [2, p. 61]), with the indicator diagram of with 2π, $\{\delta_n\}$ a bounded sequence of complex numbers, and F an entire function of the Cartwright class ([2, p. 60]) with the zero set $\{\lambda_n+\delta_n\}$. If*

$$\lim_{N\to\infty}\sup_n\frac{1}{N}\left|\Re\left(\delta_{n+1}+\delta_{n+2}+\ldots+\delta_{n+N}\right)\right|=:d<\frac{1}{4},$$

then F has the indicator diagram of with 2π, and for any $d_1>d$, h, functions $u,v\in L^\infty(\mathbb{R})$, $\|v\|_{L^\infty(\mathbb{R})}<2\pi d_1$ may be found such that

$$|F(x+ih)|^2=\exp\{u(x)+\tilde{v}(x)\}$$

for any real h such that $|h|>\sup|\Im(\lambda_n+\delta_n)|$.

For the proof of this assertion, one argues using [1, Lemma 1], and in a similar way to the proof of [1, Lemma 2] that there exists a sine type function with zeros μ_n such that, for any $d_1 > d$,

$$d_1 \Re(\mu_{n-1} - \mu_n) \le \Re(\lambda_n + \delta_n - \mu_n) \le d_1 \Re(\mu_{n+1} - \mu_n) \,.$$

Then Proposition 2 follows directly from [4, Lemmas 1, 2].

We begin to construct F putting

$$F_1(z) := z \prod_{n \in \mathbf{N}} (1 - \frac{z^2}{\nu_n^2}).$$

In view of the asymptotics (7) of ν_k and Proposition 2, $F_1(z)$ may be written in the form

$$|F_1(x+ih)|^2 = \exp\{u(x) + \tilde{v}(x)\} \tag{12}$$

with $u, v \in L^\infty(\mathbb{R})$, $\|v\|_{L^\infty(\mathbb{R})} < \varepsilon$ for any positive ε.

Further, for any fixed $\delta > 0$, we take the function $\sin \pi\delta z/2$ and for every $k \in \mathbf{K}$ find the zero $2n_k/\delta$ of this function which is the nearest to ω_k. We set

$$F_2(z) := z \prod_{n \in \mathbf{N}} (1 - \frac{z^2}{\mu_n^2}) \,,$$

where

$$\mu_n = \begin{cases} 2n/\delta & \text{for} \quad n \notin \{n_k\} \,, \\ \omega_k & \text{for} \quad n = n_k. \end{cases}$$

Using again Proposition 2 we conclude that for any $\varepsilon > 0$ function $|F_2(x+ih)|^2$ has the representation similar to (12) and the width of the indicator diagram of F_2 is 2δ.

Taking $F = F_1 F_2$ we obtain the function with indicator diagram of width $2(\pi + \delta)$ vanishing on Λ and satisfying (2). Proposition 1 is proved.

3 Proofs

3.1 Proof of Theorem 2

(i) Continuity on parameters $\{\mu_k\}$ and estimate (5) follows immediately from the representation (4). From (3) symmetry is clear if all $\{\mu_k\}$ are different. Then, by continuity, we obtain symmetry for any points in the sense that if σ is a permutation of $\{1, 2, \ldots, n\}$, then $[\mu_1, \ldots, \mu_n] = [\mu_{\sigma(1)}, \ldots, \mu_{\sigma(n)}]$.

Now we describe the structure of GDD. Let $z_1, z_2, \ldots, z_n$ be complex numbers, not necessarily different. Let us represent this set as the union of q different points, $\nu_1, \ldots, \nu_q$, with multiplicities $m_1, \ldots, m_q$; $(m_1 + m_2 + \ldots + m_q = n)$.

Lemma 4 *The GDD* $[z_1,\ldots,z_n]$ *of order* $n-1$ *is a linear combination of functions*

$$t^m e^{i\nu_k t},\ k=1,\ldots,q,\ m=0,1,\ldots,m_k-1$$

and the coefficients of the leading terms $t^{m_k-1}e^{i\nu_k t}$ *are not equal to zero.*

Proof of the lemma. The statement is clear for $n=1$. Let us suppose that it is true for GDD of the order $n-2$.

If $z_1 \neq z_n$ then, by definition, $[z_1,\ldots,z_n] = \dfrac{[z_1,\ldots,z_{n-1}]-[z_2,\ldots,z_n]}{z_1-z_n}$ and is a linear combination of the GDD's of order $n-2$. Multiplicity of the point z_n in the set $\{z_2,\ldots,z_n\}$ is more than in the set $\{z_1,\ldots,z_{n-1}\}$. Let k be such a number that $z_n=\nu_k$. Then the leading term $t^{m_k-1}e^{i\nu_k t}$ appears in $[z_2,\ldots,z_n]$ by the inductive conjecture and does not appear in $[z_1,\ldots,z_{n-1}]$.

Let $z_1 = z_n$. Then $[z_1,\ldots,z_{n-1}]$ contains the leading term $t^{m_1-2}e^{i\nu_1 t}$. After differentiation in z_1 we get the leading term $t^{m_1-1}e^{i\nu_1 t}$ with nonzero coefficient. Using symmetry of GDD relative to points $z_1, ..., z_n$, we complete the proof of the lemma. ■

We are able now, with knowledge of the structure of GDD, to continue the proof of Theorem 2.

(ii) This assertion is the consequence of the 'triangle' structure of GDD: if we add n-th point, then a GDD of the order $n-1$, in comparison with a GDD of the order $n-2$, contains either a new exponential or a term of the form $t^m e^{i\nu_p t}$ with the same frequency ν_p and larger exponent m.

(iii) If all points $\mu_1,\ldots,\mu_n$ are different, then the family of GDD's contains all exponentials $e^{i\mu_1 t},\ldots,e^{i\mu_n t}$.

(iv) Immediately follows from the definition.

(v) As well known, divided differences approximate the derivatives of the corresponding order. We use the following estimate [30].

Proposition 3 *For any* $\varepsilon>0$, $N\in I\!N$, *there exists* δ *such that for any set* $\{z_j\}_{j=1}^N$ *belonging to the disk* $D_0(\delta)$ *of a radius* δ *with center at the origin, the estimates*

$$\Big|[z_1,\ldots,z_j](t)-t^{j-1}/(j-1)!\Big| < \varepsilon,\ j=1,\ldots,N, \tag{13}$$

are valid for $t\in[-\pi N,\pi N]$.

To proceed with (v) we choose T large enough that

$$\Big\|[\mu_1,\ldots,\mu_j]\Big\|_{L^2(T,\infty)} < \varepsilon/3,\ \Big\|e^{i\mu t}t^{j-1}/(j-1)!\Big\|_{L^2(T,\infty)} < \varepsilon/3, \qquad j=1,\ldots,N, \tag{14}$$

for all $\mu, \mu_1, \ldots, \mu_j$ lying in the strip S.

Set $\varepsilon_1 = \varepsilon/3\sqrt{T}$, and let δ be small enough in order to (13) be fulfilled for such ε_1, $t \in [0,T]$ and any set $\{z_j\}_{j=1}^N \in D_0(\delta)$. Set $z_j := \mu_j - \mu$. In view of (iv), we obtain

$$\left|e^{-i\mu t}[\mu_1, \ldots, \mu_j](t) - t^{j-1}/(j-1)!\right| < \varepsilon_1, \ j = 1, \ldots, N, \ t \leq T,$$

that implies

$$\left|[\mu_1, \ldots, \mu_j](t) - e^{i\mu t}t^{j-1}/(j-1)!\right| < \varepsilon/3\sqrt{T}, \ j = 1, \ldots, N,$$

for $t \leq T$. Therefore, we have

$$\left\|[\mu_1, \ldots, \mu_j](t) - e^{i\mu t}t^{j-1}/(j-1)!\right\|_{L^2(0,T)} < \varepsilon/3, \ j = 1, \ldots, N.$$

Taking into account (14) we obtain the statement (v). ∎

3.2 Proof of the Theorem 3.

(i) From Lemma 2 it follows that the family $\{\mathcal{L}^{(p)}\} := \{\mathcal{L}^{(p)}(r)\}$ forms a Riesz basis in the closure of its span (i.e. an $\mathcal{L}$-basis) in $L^2(0,\infty)$. Introduce the projector P_T from this closure *into* $L^2(0,T)$. Clearly, $\{\mathcal{L}^{(p)}\}$ forms a Riesz basis in $L^2(0,T)$ if and only if P_T is an isomorphism *onto* $L^2(0,T)$. This takes a place if and only if the generating function of the family $\mathcal{E}(\Lambda)$ satisfies the Helson-Szegö condition (2) (see [18]; [2], Theorems II.3.14, II.3.17).

The $\mathcal{L}$-basis property of the subspaces $\mathcal{L}^{(p)}$ means that for any finite number of functions $\psi_p \in \mathcal{L}^{(p)}$ we have the estimates:

$$\|\sum_p a_p\psi_p\|^2_{L^2(0,\infty)} \asymp \sum_p |a_p|^2||\psi_p||^2_{L^2(0,\infty)}. \tag{15}$$

This relation means that there exist two positive constants c and C which do not depend on a sequence $\{a_p\}$ such that

$$c\|\sum_p a_p\psi_p\|^2_{L^2(0,\infty)} \leq \sum_p |a_p|^2||\psi_p||^2_{L^2(0,\infty)} \leq C\|\sum_p a_p\psi_p\|^2_{L^2(0,\infty)}.$$

In each subspace $\mathcal{L}^{(p)}$ we choose the family of GDD's corresponding to $\lambda_{j,p} \in \Lambda^{(p)}$: $\varphi_j^{(p)} := [\lambda_{1,p}, \lambda_{2,p}, \ldots, \lambda_{j,p}]$. $j = 1, \ldots, \mathcal{N}^{(p)}$. In view of Theorem 2, this family forms a basis in $\mathcal{L}^{(p)}$.

Let us expand ψ_p in the basis $\{\varphi_j^{(p)}\}_{j=1}^{j=\mathcal{N}^{(p)}}$. Then statement (i) of the theorem is equivalent to the estimates

$$\|\sum_{p,j} a_{p,j}\varphi_j^{(p)}\|^2_{L^2(0,\infty)} \asymp \sum_{p,j} |a_{p,j}|^2. \tag{16}$$

Taking into account (15), we see that (16) is equivalent to

$$\|\sum_{j} a_{j}\varphi_j^{(p)}\|^2_{L^2(0,\infty)} \asymp \sum_{j} |a_{j}|^2, \text{uniformly in } p. \tag{17}$$

We introduce the $\mathcal{N}^{(p)} \times \mathcal{N}^{(p)}$ Gram matrices $\Gamma^{(p)}$ corresponding to the families $\mathcal{E}^{(p)}$,

$$\Gamma^{(p)} := \left\{(\varphi_k^{(p)}, \varphi_j^{(p)})_{L^2(0,\infty)}\right\}_{k,j}.$$

For an $\mathcal{N}^{(p)}$-dimensional vector a we have

$$\|\sum_{j} a_{j}\varphi_j^{(p)}\|^2_{L^2(0,\infty)} = \langle \Gamma^{(p)} a, a\rangle,$$

where $\langle\cdot,\cdot\rangle$ is the scalar product in $\mathbb{R}^{\mathcal{N}^{(p)}}$.

In terms of the Gram matrices, (17) is true if and only if matrices $\Gamma^{(p)}$ and their inverses are bounded uniformly in p.

Lemma 5 *Let different complex points $\mu_1, \mu_2, \ldots, \mu_n$ lie in a disk $D_\mu(R) \subset \mathbb{C}_+$ of radius R with the center μ, $\mu \in S$; $\varphi_1, \varphi_2, \ldots, \varphi_n$ are corresponding GDD's, and Γ is the Gram matrix of this family in $L^2(0,\infty)$. Then the norms $\langle\langle\Gamma\rangle\rangle$ and $\langle\langle\Gamma^{(-1)}\rangle\rangle$ are estimated from above by constants depending only on R and n.*

PROOF: By Theorem 2, functions $\varphi_1(t), \varphi_2(t), \ldots, \varphi_n(t)$ are estimated above by constants depending only on R, n. Therefore the entries of Γ are estimated by *const* (R, n) and, then, the estimate of Γ from above is obtained.

We shall prove the estimate for the inverse matrices by contradiction. Let us fix a disk and assume that for arbitrary $\varepsilon > 0$ there exist different points $\mu_1^{(\varepsilon)}, \mu_2^{(\varepsilon)}, \ldots, \mu_n^{(\varepsilon)}$ lying in the disk, and normalized n-dimensional vectors $a^{(\varepsilon)}$ such that for the corresponding Gram matrix $\Gamma^{(\varepsilon)}$ we have

$$\langle \Gamma^{(\varepsilon)} a^{(\varepsilon)}, a^{(\varepsilon)}\rangle \le \varepsilon. \tag{18}$$

Using compactness arguments we can choose a sequence $\varepsilon_n \to 0$ such that

$$a^{(\varepsilon_n)} \to a^0,\ \mu_j^{(\varepsilon_n)} \to \mu_j^0.$$

as $n \to \infty$. Then the Gram matrix tends to the Gram matrix Γ^0 for the limit family and from (18) we see that

$$\langle \Gamma^0 a^0, a^0 \rangle = 0.$$

Then $\sum a_j^0 \varphi_j^0 = 0$ and the limit family of GDD's is linearly dependent that contradicts to Theorem 2.

Thus, we have proved that the norm of $\Gamma^{(-1)}$ is estimated from above by constants depending only on R, n, and μ. In view of Theorem 2(iv) the constant depends actually not on μ, but on $\Im\mu$. Indeed, translation of the disk on a real number, $\mu \mapsto \mu + x$, does not change the Gram matrix. Since $\alpha \le \Im\mu \le \beta$, the norm of $\Gamma^{(-1)}$ is estimated uniformly in $\mu \in S$. ■

From this lemma it follows that all Gram matrices Γ_p, $\Gamma_p^{(-1)}$ are bounded uniformly in p. (It was supposed in the lemma that all points $\mu_1, \mu_2, \ldots, \mu_n$ are distinct. For the case of multiple points we use the continuity of DD on parameters.) The assertion (i) of the Theorem 3 is proved.

(ii) We need to estimate the Gram matrices Γ for the exponential family $\{e^{i\mu_1 t}, \ldots, e^{i\mu_n t}\}$ in $L^2(0,T)$ from below, where $\{\mu_j\}$ is a fixed set $\Lambda^{(p)}$ and $n = \mathcal{N}^{(p)}$. Since the projector P_T is an isomorphism (see (i)), we may do it for exponentials on the positive semiaxis, i.e., in $L^2(0,\infty)$. Denote by $e_j(t)$ the normalized exponentials

$$e_j(t) := \frac{1}{\sqrt{2\Im\mu_j}} e^{i\mu_j t}.$$

For the Gram matrices Γ_0, corresponding to the normalized exponentials, we have

$$\Gamma = \operatorname{diag}[\sqrt{2\Im\mu_j}]\, \Gamma_0 \operatorname{diag}[\sqrt{2\Im\mu_j}],$$

and we can estimate Γ_0 instead Γ, since $|\Im\mu_j| \asymp 1$. It can be easily shown that the inverse matrix is the Gram matrix for the biorthogonal family $e_j'(t)$ and

$$(\Gamma_0)_{jj}^{(-1)} = \|e_j'\|^2 = \prod_{k \neq j} \left| \frac{\mu_k - \bar{\mu}_j}{\mu_k - \mu_j} \right|^2$$

(see [25, 2]). Elementary calculations give

$$(\Gamma_0)_{jj}^{(-1)} = \|e_j'(t)\|^2 \prec \delta_p^{-2(n-1)}$$

Then

$$|(\Gamma_0)_{jk}^{(-1)}| = (e_j', e_k') \le \|e_j'\| \|e_k'\| \prec \delta_p^{-2(n-1)}$$

and so,

$$\langle\langle\Gamma_0^{(-1)}\rangle\rangle \prec \delta_p^{-2(n-1)}.$$

The theorem is proved. ■

3.3 Proof of Theorem 4.

As well known, the family $\{t^n\}$, $n = 0, 1, \ldots$, of powers is not minimal in $L^2(0, T)$ for any T. Moreover, direct calculations for $\varphi_j^0 := e^{i\mu t} t^{j-1}/(j-1)!$ give

$$\text{angle}_{\,L^2(0,\infty)}(\varphi_m^0, \varphi_{m+1}^0) \to 0, \quad m \to \infty,$$

uniformly in $\mu \in S$. Since Λ is not a relatively uniformly discrete sequence, for any $\delta > 0$, $m \in \mathbb{N}$ we are able to find a disk $D_\mu(\delta)$ in the strip S, which contains m points of Λ. In view of Theorem 2(v), GDD corresponding to these points are ε–close to 'unperturbed' functions φ_j^0. This proves the assertion of Theorem 4.

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